

EFFECTIVE BOUNDS FOR THE NUMBER OF MINIMAL MODEL PROGRAMS OF A SMOOTH THREEFOLD

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ABSTRACT. We prove that the number of minimal model programs of a smooth projective threefold of positive Kodaira dimension obtained performing first a series of divisorial contractions and then a series of flips can be bounded by a constant that depends only on the Picard number. We obtain as a corollary that if the Picard number is equal to three, the number of minimal model programs is at most three.

1. INTRODUCTION

Establishing the existence of minimal models is one of the first steps towards the birational classification of smooth projective varieties. Moreover, starting from dimension three, minimal models are known to be non-unique, leading to some natural questions such as: does a variety admit a finite number of minimal models? And if yes, can we fix some parameters to bound this number?

Thanks to the groundbreaking result [BCHM10], we know that varieties of general type admit a finite number of minimal models. For varieties of non-general type this number can be infinite, see [Rei83, Example 6.8]. However, it is conjectured that the number of minimal models up to isomorphism is always finite. This is known for threefolds of positive Kodaira dimension [Kaw97].

In [MST16] it is proved that it is possible to bound the number of minimal models of a smooth variety of general type and bounded volume. Moreover, in dimension three Cascini and Tasin [CT14] bounded the volume using the Betti numbers. This result can be used to show that the number of minimal models of a threefold of general type can be bounded using topological data, solving a conjecture of Cascini and Lazić [CL14].

In this note we address a closely related, although different, question. In the case of a smooth threefold of positive Kodaira dimension we bound in an effective way a subset, obtained under some technical assumptions, of the minimal model programs of X . Specifically,

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we bound how many are the possible series of K_X -negative birational contractions starting from X .

Our main theorem is the following.

Theorem 1.1. *Let X be a smooth projective threefold of positive Kodaira dimension. Then the number of minimal model programs of X that can be obtained performing first a series of divisorial contractions followed by a series of flips is at most $\max\{1, 3^{\rho(X)-2}[(\rho(X) - 2)!]^2\}$, where $\rho(X)$ is the Picard number of X .*

Theorem 1.1 reduces quickly to finding a bound for the number of possible flipping curves passing through a terminal singularity, see [KM98, Definition 2.34]. Note that in the case of a smooth surface S of positive Kodaira dimension, the minimal model is unique [KM98, Definition 1.30] and, therefore, we cannot have two (-1) -curves E_1 and E_2 passing through the same point. Indeed, they both should be contracted before reaching the minimal model of S , but the contraction of C_1 transforms C_2 in a curve with non-negative selfintersection, and so impossible to contract.

The assumption on the order of flips and divisorial contractions comes from the fact that we cannot control the number of flipping curves contained in a divisor that is later contracted in the MMP. In the non-general type case there exists an example of a threefold X where this number of curves is infinite, hence producing a new example of a variety with non-negative Kodaira dimension and an infinite number of K_X -negative extremal rays, see [Les15, Theorem 7.1]. In the general-type case this cannot happen because of the finiteness statement proved in [BCHM10], and there is still hope to bound the total number of minimal model programs using the topology of the variety.

Theorem 1.1 is a technical result, in the sense that in general we cannot impose geometric conditions on X so that the hypothesis is satisfied. However, using Theorem 1.1 we can obtain some effective bounds in the case of Picard number equal to three.

Theorem 1.2. *Let X be a smooth projective threefold of positive Kodaira dimension of Picard number $\rho(X) = 3$, then the number of minimal model programs of X is at most 3.*

The paper is organized as follows: in Section 3 we collect some preliminary notions, mainly about the MMP in dimension three. The reader in need of more details should refer to [KM98]. In Section 4 we prove Theorem 1.1 and Theorem 1.2. We conclude with a possible strategy to bound the number of minimal model programs in the case of Picard number equal to four, see Section 4.1.1.

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3. PRELIMINARY RESULTS

We will always refer to an MMP for X as a series of K_X -negative birational contractions; in this context, a minimal model for X is just an outcome of an MMP for X .

Definition 3.1. Let $f: X \dashrightarrow Y$ be a birational map, we recall that the exceptional locus of f , that we denote with $\text{Exc}(f)$, is the locus of X where f is not an isomorphism.

3.1. The Picard number. Let X be a normal variety. Two Cartier divisors D_1 and D_2 on X are numerically equivalent, $D_1 \equiv D_2$, if they have the same degree on every curve on X , i.e. if $D_1 \cdot C = D_2 \cdot C$ for each curve C in X . The quotient of the group of Cartier divisors modulo this equivalence relation is denoted by $N^1(X)$.

We can also define $N^1(X)$ as the subspace of cohomology $H^2(X, \mathbb{Z})$ spanned by divisors. We write $N^1(X)_{\mathbb{Q}} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. $N^1(X)_{\mathbb{Q}}$ is a finite dimensional vector space.

Definition 3.2. We define $\rho(X) := \dim_{\mathbb{Q}} N^1(X)_{\mathbb{Q}}$ and we call it the Picard number of X .

We remark that $\rho(X) \leq b_2$, the second Betti number of X , that depends only on topological information of X .

Similarly, two 1-cycles C_1 and C_2 are numerically equivalent if they have the same intersection number with any Cartier divisor. We call $N_1(X)$ the quotient group and we write $N_1(X)_{\mathbb{Q}} := N_1(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. We can also see $N_1(X)$ as the subspace of homology $H_2(X, \mathbb{Z})$ spanned by algebraic curves. See for details [Deb13, Section 1.4].

We will also use the following definition.

Definition 3.3. Let E be an irreducible divisor contained in a projective variety X . We consider the following map

$$\psi : N_1(E)_{\mathbb{Q}} \longrightarrow N_1(X)_{\mathbb{Q}}$$

and we denote with $N_1(E|X)_{\mathbb{Q}} := \psi(N_1(E)_{\mathbb{Q}}) \subseteq N_1(X)_{\mathbb{Q}}$, and $\rho(E|X) := \dim N_1(E|X)_{\mathbb{Q}}$. Note that $\ker \psi$ might be not empty.

3.2. The difficulty. In dimension three, the existence and termination of flips was proved by Mori and Shokurov. A key ingredient for the proof of termination is the so called *difficulty* of X , introduced by Shokurov.

Definition 3.4. [Sho86, Definition 2.15], [KM98, Definition 6.20] Let X be a projective threefold, then the difficulty of X is defined as follows

$$d(X) := \#\{E \text{ prime divisor} \mid a(E, X) < 1, E \text{ is exceptional over } X\},$$

where $a(E, X)$ is the discrepancy of E with respect of X .

Remark 3.5. Note that the difficulty always goes down under a flip, and if X is smooth, then $d(X) = 0$ and we cannot have any flips. See [KM98, Lemma 3.38].

We also recall that minimal models are connected by a sequence of flops, i.e. by an isomorphism in codimension one, [Kol89, Theorem 4.9].

For the definitions of divisorial contraction, flip and flop we refer to [KM98, Proposition 2.5], [KM98, Definition 6.10]. We just recall that under a divisorial contraction $f: X \dashrightarrow X'$ the Picard number drops by one, i.e. $\rho(X') = \rho(X) - 1$; if f is a flip instead, the Picard number is stable: $\rho(X') = \rho(X)$.

4. PROOF OF THEOREM 1.1 AND THEOREM 1.2

In this section, we prove Theorem 1.1 and Theorem 1.2. The strategy of the proof is to first bound the number of steps of the MMP and then count how many are the possible divisorial contractions to a point, to a curve and flips at a certain step.

Our starting point is a smooth projective threefold X of positive Kodaira dimension. As we recalled in Remark 3.5, this means that the difficulty of X has to be zero and no flips are possible. Then, the first operation of the MMP for X has to be a divisorial contraction. Let us assume that $\rho(X) = 2$ (if $\rho(X) = 1$ no contractions are possible).

Lemma 4.1. *Let X be a smooth projective threefold of positive Kodaira dimension such that $\rho(X) = 2$. Then there is only a unique MMP for X .*

Proof. After one divisorial contraction ϕ' we reach a variety X' with Picard number equal to one and we stop. If there were an other divisorial contraction ϕ'' onto a different variety X'' , we would have a sequence of flops connecting the minimal varieties X' and X'' , see [Kol89, Theorem 4.9], but that is impossible since $\rho(X') = 1$ and so no contractions are possible. \square

Remark 4.2. Therefore, we can always assume $\rho(X) \geq 3$ and that $\rho(X') \geq 2$ for X' a minimal model for X , since otherwise we have only one possible MMP.

Let us proceed now with the bound for the number of steps. It is a calculation that follows from the termination of flips in dimension three, see [CZ14, Lemma 3.1].

Lemma 4.3. *Let X be a smooth projective threefold of positive Kodaira dimension such that $\rho(X) \geq 3$. Let X' be the outcome of a MMP for X , we suppose in addition that $\rho(X') \geq 2$. Let \mathcal{S} be the total number of steps of a minimal model program of X . Then \mathcal{S} is at most $2(\rho(X) - 2)$.*

Proof. We denote with \mathcal{D}_C the total number of divisorial contractions and with \mathcal{F} the total number of flips. Let X' be the outcome of an MMP for X .

Clearly $\mathcal{S} = \mathcal{D}_C + \mathcal{F}$. Under a divisorial contraction the Picard number drops by one. Hence, $\mathcal{D}_C = \rho(X) - \rho(X') \leq \rho(X) - 2$. To conclude the proof, we claim that $\mathcal{F} \leq \mathcal{D}_C$. Under a flip, the Picard number is stable and we need to consider the difficulty $d(X)$, see Definition 3.4. If X is smooth, $d(X) = 0$ and no flips are possible, see Remark 3.5. Moreover, if $X_{i-1} \rightarrow X_i$ is a divisorial contraction, then

$$d(X_i) \leq d(X_{i-1}) + 1,$$

since the contraction might have created some singularities. Otherwise, if $X_{i-1} \dashrightarrow X_i$ is a flip, then

$$d(X_i) \leq d(X_{i-1}) - 1,$$

because flips strictly improve the singularities (see [KM98, Definition 6.20, Lemma 3.38]). We conclude that in order to have a flip, we first need to have had a divisorial contraction. Thus, $\mathcal{F} \leq \mathcal{D}_C$ and $\mathcal{S} \leq 2(\rho(X) - 2)$. \square

Let X be a smooth threefold of positive Kodaira dimension, satisfying all the assumptions of Theorem 1.1 and Lemma 4.3. Let X' be the outcome of an MMP ϕ' for X composed by a series of divisorial contractions followed by a series of flip. Then we can represent ϕ' in the following way.

$$\phi': X = \underbrace{X_0 \rightarrow X_1 \rightarrow \cdots \rightarrow X_i}_{\text{divisorial contractions}} = \underbrace{X^0 \dashrightarrow X^1 \dashrightarrow \cdots \dashrightarrow X^j}_{\text{flips}} = X'.$$

We will always indicate with X_i , $0 \leq i \leq \rho(X) - 2$, a step in the minimal model program for X that can be reached from X with a series of divisorial contractions; with X^j , $0 \leq j \leq \rho(X) - 2$, a step in the minimal model program for X from which we can reach the minimal model X' , with a series of flips.

We now proceed to bound the number of divisorial contractions to a point.

Lemma 4.4. *Let $X = X_0$ be a smooth projective threefold of positive Kodaira dimension satisfying the assumptions of Lemma 4.3 and Theorem 1.1. Let X_i be a step in the minimal model program for X , $1 \leq i \leq \rho(X) - 2$. The number of ways to go from X_i to the following step with a divisorial contraction to a point is at most $\rho(X) - 2 - i$.*

Proof. We divide the proof into steps.

Step 1. We consider the last divisorial contraction.

$$\psi: X_{i-1} \rightarrow X_i = X^0$$

and we claim that there is a unique choice of divisor E that can be contracted to a point by ψ . In particular, ψ is uniquely determined by X_{i-1} .

Proof of Step 1. Let us assume by contradiction that there are two distinct divisors E_1 and E_2 contained in X_{i-1} and that they can be both contracted to a point.

$$\begin{array}{ccc} X_{i-1} & \xrightarrow{\psi^1} & X'_i \\ & \searrow \psi^2 & \\ & & X''_i \end{array}$$

We denote with ψ^1 the contraction of E_1 and with ψ^2 the contraction of E_2 . But then, X'_i contains $\psi^1(E_2)$ and X''_i contains $\psi^2(E_1)$, as divisors. Since X'_i and X''_i are followed only by flips, $\psi^1(E_2)$ and $\psi^2(E_1)$ are not contracted and survive until the minimal models.

$$\begin{array}{ccccc} X_{i-1} & \xrightarrow{\psi^1} & X'_i & \text{---} & X' \\ & \searrow \psi^2 & & & \downarrow \eta_{12} \\ & & X''_i & \text{---} & X'' \end{array}$$

We have that $X' \supseteq \psi_1(E_2)$ and $X'' \supseteq \psi_2(E_1)$, (where by abuse of notation with still indicate with $\psi_1(E_2)$ and $\psi_2(E_1)$ the images of the divisors through the series of flips). However, X' and X'' are minimal models and are, therefore, connected by a sequence of flops η_{12} , i.e. an isomorphism in codimension one, [Kol89, Theorem 4.9]. We reach a contradiction. \square

Step 2.

Let us now consider the preceding step. In this case we have at least two divisors E_1 and E_2 that can be contracted to a point.

$$\begin{array}{ccccc} X_{i-2} & \xrightarrow{\psi'_{i-1}} & X'_{i-1} & \xrightarrow{\psi'_i} & X'_i \\ & \searrow \psi''_{i-1} & & & \\ & & X''_{i-1} & \xrightarrow{\psi''_i} & X''_i \end{array}$$

There are just two possibilities, we can either contract first E_1 with ψ'_{i-1} and then E_2 with ψ'_i or we can invert the order and contract first E_2 with ψ''_{i-1} and then E_1 with ψ''_i . We claim that there are no more possible divisors that can be contracted into a point.

Proof of Step 2. Let us assume by contradiction that there exists an other divisor E_3 , such that E_3 is distinct from E_1 and E_2 and that can be contracted into a point by a divisorial contraction that we call $\psi'''_{i-1}: X_{i-2} \rightarrow X'''_{i-1}$. Since E_3 is not contracted by ψ'_{i-1} and ψ'_i , $\psi'_i(\psi'_{i-1}(E_3))$ is contained in X'_i and so also in the minimal model X' because X'_i is followed just by flips. If we consider instead the minimal model X''' that follows X'''_{i-1} , in particular this means that X''' is obtained contracting $K_{X'''_{i-1}}$ -negative extremal rays, X''' is not going to contain the image of E_3 . But since X' and X''' are minimal models and so connected by an isomorphism in codimension one, we reach a contradiction. \square

Step 3. At the step X_i , for $0 \leq i \leq \rho(X) - 2$, the number of ways to go from X_i to the following step with a divisorial contraction to a point is at most $\rho(X) - 2 - i$.

Proof of Step 3. We can iterate the argument of Step 2 in the case of a series of divisorial contractions. Let X_i be a step of the minimal model for X , there are at least $\rho(X) - 2 - i$ choices for a divisor to be contracted to a point. If there were at least $\rho(X) - 1 - i$ choices, there would be a divisor E that survives until we reach the minimal model X' . But since at the step X_i the divisor E can be contracted, there exists another MMP ϕ'' such that ϕ'' contracts E and again we reach a contradiction since minimal models are isomorphic in codimension one. We conclude that the number of possible contractions to a point at the step X_i is at most $\rho(X) - 2 - i$. \square

This conclude the proof of the lemma. \square

We want now to count how many are the possible choices for a divisorial contraction to a curve.

Lemma 4.5. *Let $X = X_0$ be a smooth projective threefold of positive Kodaira dimension satisfying the assumptions of Lemma 4.3 and Theorem 1.1. Let X_i be one step in the minimal model program for X , $0 \leq i \leq \rho(X) - 2$. The number of ways to go from X_i to the following step with a divisorial contraction to a curve is at most $2(\rho(X) - 2 - i)$.*

Proof. Again we divide the proof into steps.

Step 1. We consider the last divisorial contraction.

$$\psi: X_{i-1} \rightarrow X_i = X^0$$

and we claim that there are at most two divisorial contractions to a curve.

Proof of Step 1. Proceeding as we did in Step 1 of Lemma 4.4, we can prove that there is a unique choice for a divisor E to be contracted. Then we need to understand in how many ways the divisor E can be contracted into a curve C . We know that $\rho(C) = 1$ because two

divisors on a curve are numerically equivalent if they have the same degree, then we obtain that $\rho(E|X) = 2$, see Definition 3.3, because $\rho(X_{i-1}) - \rho(X_i) = 1$, since ψ_i is a divisorial contraction. We then obtain two possible contractions: ψ^1 that contracts E into a curve C_1 and ψ^2 that contracts E into a curve C_2 .

$$\begin{array}{ccc} X_{i-1} & \xrightarrow{\psi^1} & X'_i \\ & \searrow \psi^2 & \downarrow \eta_{12} \\ & & X''_i \end{array}$$

□

Step 2. Let X_i be one step in the minimal model program for X , $0 \leq i \leq \rho(X) - 2$. The number of ways to go from X_i to the following step with a divisorial contraction to a curve is at most $2(\rho(X) - 2 - i)$.

Proof of Step 2. Let X_i be a step of the minimal model for X , there are at least $\rho(X) - 2 - i$ choices for a divisor to be contracted. If there were at least $\rho(X) - 1 - i$ choices, there would be a divisor E that survives until we reach the minimal model X' . But since at the step X_i the divisor E can be contracted, there exists another MMP ϕ'' such that ϕ'' contracts E and again we reach a contradiction since minimal models are isomorphic in codimension one. Moreover, each of this divisor can be contracted in at most two different ways as we explained in Step 1. We conclude that the number of possible contractions to a point at the step X_i is at most $2(\rho(X) - 2 - i)$. □

This conclude the proof of the Lemma. □

Remark 4.6. The simplest example of the situation in Step 1 of Lemma 4.5 is the case of Atiyah's flop, see for instance [HM10, Example 1.16], where $E \cong \mathbb{P}^1 \times \mathbb{P}^1$ and the map η_{12} between X'_i and X''_i is the flop that sends $C_1 \cong \mathbb{P}^1$ into $C_2 \cong \mathbb{P}^1$.

In conclusion, we have obtained the following lemma.

Lemma 4.7. *Let $X = X_0$ be a smooth projective threefold of positive Kodaira dimension satisfying the assumptions of Lemma 4.3 and Theorem 1.1. Let X_i be a step in the minimal model program for X , $0 \leq i \leq \rho(X) - 2$. The number of ways to go from X_i to the following step with a divisorial contraction is at most $3(\rho(X) - 2 - i)$.*

The difficult part is to bound the number of possible ways to go from one step to the following with a flip.

Proposition 4.8. *Let $X = X_0$ be a smooth projective threefold of positive Kodaira dimension satisfying the assumptions of Lemma 4.3 and Theorem 1.1. Let X^j be a step in the minimal model program for*

X , $0 \leq j \leq \rho(X) - 2$. Then the number of ways to go from X^j to the following step with a flip is at most $\rho(X) - 2 - j$.

Proof. We divide the proof into steps.

Step 1. We consider the last flip in the minimal model program for X .

$$\psi : X^{j-1} \dashrightarrow X^j = X'$$

where $0 \leq j \leq \rho(X) - 2$. and we claim that there is a unique choice of curve ξ that can be flipped by ψ .

Proof of Step 1. Assume by contradiction that there are two possible flips into two distinct minimal models X' and X'' .

$$\begin{array}{ccc} X^{j-1} & \xrightarrow{\psi^1} & X' \\ & \searrow \psi^2 & \downarrow \eta_{12} \\ & & X'' \end{array}$$

Where we denoted by η_{12} the sequence of flops connecting the two minimal models X' and X'' (see [Kol89, Theorem 4.9]) and by ξ^2 the curve that is flipped by ψ^2 . Now, thanks to the Abundance Theorem [Kol92], there exists an integer m such that $|mK_{X'}|$ is free. Therefore, we can choose a general surface

$$(1) \quad S \in |mK_{X'}|$$

such that it does not contain any irreducible components of $\text{Exc}(\eta_{12})$. Then let $S_0 := (\psi_1^{-1})_*(S)$ be the strict transform of S , since flips are isomorphisms in codimension one, $S_0 \in |mK_{X^{j-1}}|$. We are assuming that there exists another flipping curve ξ_2 such that $\xi_1 \neq \xi_2$ and so $\xi_2 \not\subseteq \text{Exc}(\psi^1)$. Since ξ_2 is a flipping curve, $K_{X^{j-1}} \cdot \xi_2 < 0$, and so $\xi_2 \subseteq S_0$. Now we consider the restriction of ψ^1 to S_0

$$g := \psi|_{S_0}^1 : S_0 \dashrightarrow S$$

and since $\text{Exc}(g) \subseteq \text{Exc}(\psi^1) \cap S_0$, we obtain that $\xi_2 \not\subseteq \text{Exc}(g)$ and so $\psi^1(\xi_2) \subseteq S$. This is a contradiction, because $\psi^1(\xi_2)$ is flopped by η_{12} into $\psi^2(\xi_1)$ but S was chosen in such a way that it does not contain any irreducible components of $\text{Exc}(\eta_{12})$. \square

Step 2. Let X^j be a step in the minimal model program for X , $0 \leq j \leq \rho(X) - 2$. Then the number of ways to go from X^j to the following step with a flip is at most $\rho(X) - 2 - j$.

Proof. We can iterate the argument of Step 1 in the case of a series of flips. Let X_i be a step of the minimal model for X , there are at least $\rho(X) - 2 - i$ choices for possible flipping curves. If there were at least $\rho(X) - 1 - i$ choices, there would be a curve ξ that survives until we reach the minimal model X' and so is contained in the surface S chosen in (1). But since at the step X_i the curve ξ is a flipping

curve, there exists another MMP ϕ'' that flips ξ and again we reach a contradiction because ξ would be contained in $\text{Exc}(\eta_{12})$. We conclude that the number of possible flips at the step X_i is at most $\rho(X) - 2 - i$. \square

This conclude the proof of the lemma. \square

Now we have all the ingredients to prove Theorem 1.1.

Proof of Theorem 1.1. Let X' be the outcome of an MMP ϕ' for X . If $\rho(X) = 2$ or $\rho(X') = 1$, then X' has to be unique, see Remark 4.2, Lemma 4.1. Otherwise, we are in the condition of Lemma 4.3. Then the proof is elementary combinatorics. After the sequence of divisorial contractions, using Lemma 4.7 we have $3^{(\rho(X)-2)}(\rho(X)-2)!$ end points. Then after the sequence of flips, thanks to Proposition 4.8, we have the final number of minimal model programs: $3^{(\rho(X)-2)}[(\rho(X)-2)!]^2$. \square

4.1. Bounds for low Picard number. In this section we apply Theorem 1.1 to obtain explicit bounds for the number of minimal model programs in the case of threefolds of low Picard rank.

We will use explanatory graphs for the MMP that can be read as follows: $X^{a,b}$ denotes a variety X such that $\rho(X) = a$ and $d(X) = b$, see Definition 3.2 and 3.4. Divisorial contractions are going to be denoted by continue arrows, and flips by dash arrows.

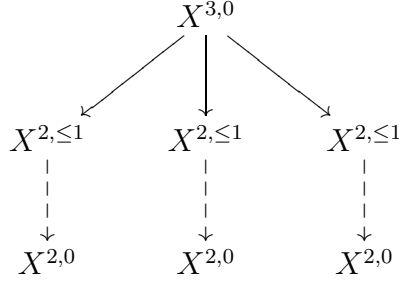
Proof of Theorem 1.2. Let X be a smooth projective threefold of positive Kodaira dimension, such that $\rho(X) = 3$. We are in the conditions to apply Theorem 1.1. Indeed, let X' be an outcome of an MMP ϕ' for X . We can assume that $\rho(X') \geq 2$, because otherwise X' is unique, see Remark 4.2. In this case the graph of the MMP for X is extremely simple: the first operation is a divisorial contraction

$$X^{3,0} \rightarrow X^{2,\leq 1}.$$

Then we can only have a flip

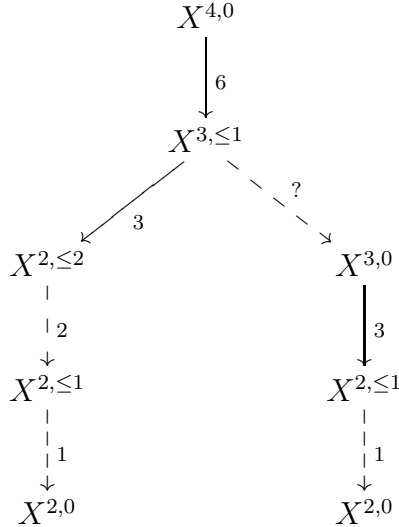
$$X^{2,\leq 1} \dashrightarrow X^{2,0}$$

and we reach an end point. Hence, the condition of Theorem 1.1 are satisfied and we can conclude that the number of minimal model programs for X are at most three. The following is an explicit graph of the MMP for X in this case,



where after the first divisorial contraction we reach varieties characterized by Picard number equal to two and difficulty less or equal one, and then after the flip we stopped having reached varieties with difficulty equal to zero and Picard number equal to two. \square

4.1.1. *Strategy for $\rho(X) = 4$.* In this last section we present a strategy to find an explicit bound in the case of Picard number equal to four and we highlight the main difficulties. Let X be a smooth projective threefold of positive Kodaira dimension such that $\rho(X) = 4$. Again we assume that if X' is the outcome of an MMP ϕ' for X , then $\rho(X') \geq 2$. The situation is more complicated. The graph in this case can be described in the following way



The numbers at the right of the arrows represent the valence of the arrow in the graph, i.e. in how many ways can be realized the operation corresponding to that arrow. The valence is computed applying all the results of the previous section. In order to count the number of end points, we need to compute the valence of the missing arrow that corresponds to a flip followed by a divisorial contraction, and so we can not conclude using Proposition 4.8.

Let us fix some notation

$$X^{3,\leq 1} \xrightarrow{\phi_1} X^{3,0} \xrightarrow{\phi_2} X^{2,\leq 1}$$

where ϕ_1 is a flip and ϕ_2 is a divisorial contractions.

- Let $\xi \subseteq X^{3,\leq 1}$ be the flipping curve.
- Let $\xi^+ \subseteq X^{3,0}$ be the flipped curve.
- Let $E \subseteq X^{3,\leq 1}$ be the divisor that is contracted by ϕ_2 .
- Let $E^+ \subseteq X^{3,0}$ be the image of the divisor through the flip ϕ_1 .

Lemma 4.9. *If $\xi \not\subseteq E$, then the flip can be realized in at most two ways.*

Proof. We consider the surface $S \in |mK_{X^{2,0}}|$, for $m > 0$ as defined in (1), and its strict transform $S' \subseteq X^{2,\leq 1}$, since flips do not change divisors $S' \in |mK_{X^{2,\leq 1}}|$. We have that $mK_{X^{3,\leq 1}} = mK_{X^{2,\leq 1}} + \beta E$, where $\beta > 0$, because $X^{2,\leq 1}$ is terminal, if we choose S_1 a general element in $|mK_{X^{3,\leq 1}}|$, then $S_1 = (\phi_1^{-1})_* S' + \beta E$. Since $\xi \cdot K_{X^{3,\leq 1}} < 0$, because ξ is a flipping curve, $\xi \subseteq S_1$. But we are assuming that $\xi \not\subseteq E$, so this forces $\xi \in (\phi_1^{-1})_* S'$, but then we can conclude thanks to the same argument of Proposition 4.8. \square

The major problem if $\xi \subseteq E$ is that E is not normal. Indeed, since the discrepancies are increasing under a flip, thanks to [KM98, Proposition 6.21] the multiplicity of E along ξ is greater than one. Then E is singular along ξ and so not normal, since E is a surface.

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